Solutions of Week 1

MATH 2040B

September 21, 2020

1 Solutions

1. Let $\text{Mat}_{2\times2}$ ($\mathbb{C}$) be the set of 2×2 complex matrices, and $u_2 \subset \text{Mat}_{2\times2}(\mathbb{C})$ be the subset of skew symmetric matrices, i.e.

$$u_2 = \{A \in \text{Mat}_{2\times2}(\mathbb{C}) : A^* + A = 0\}$$

(a) Show that $\text{Mat}_{2\times2}(\mathbb{C})$ with the usual matrix addition and scalar multiplication forms a complex vector space. Is it also a real vector space?

(b) Show that u_2 is a real subspace of $\text{Mat}_{2\times2}(\mathbb{C})$, is it also a complex subspace of $\text{Mat}_{2\times2}(\mathbb{C})$?

Solutions:

(a) Firstly, Let $V = \text{Mat}_{2\times2}(\mathbb{C})$, and then we should show that the space V equipped with usual matrix addition and scalar multiplication is closed. For $a \in F$ and $\vec{x}, \vec{y} \in V$, we have to verify:

$$(\vec{x}, \vec{y}) \mapsto \vec{x} + \vec{y} \in V$$
$$(a, \vec{x}) \mapsto a\vec{x} \in V$$

Secondly, we have to verify these two operations satisfy the eight conditions for vector spaces (Lecture notes 1).

Take the condition (vs 1) for example, for $\forall$

$$\vec{x} = \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} \text{and} \quad \vec{y} = \begin{bmatrix} u_2 & v_2 \\ w_2 & z_2 \end{bmatrix} \in V$$

and for $\forall a \in \mathbb{C}$

$$\begin{aligned}
\vec{x}+\vec{y} &= \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} + \begin{bmatrix} u_2 & v_2 \\ w_2 & z_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1+u_2 & v_1+v_2 \\ w_1+w_2 & z_1+z_2 \end{bmatrix} \\
&= \begin{bmatrix} u_2+u_1 & v_2+v_1 \\ w_2+w_1 & z_2+z_1 \end{bmatrix} \\
&= \begin{bmatrix} u_2 & v_2 \\ w_2 & z_2 \end{bmatrix} + \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} \\
&= \vec{y}+\vec{x}
\end{aligned}$$

where we have used that the elements from complex field satisfy the commutative rule.

We omit a detailed proof here, but remember to write it step by step even though it's simple. The conclusion is that V is complex and real vector space.

(b) To verify u_2 is a real subspace of V, we should consider the 3 conditions:

(1) $\vec{0}_V \in W$.

(2) $\vec{x}+\vec{y} \in W, \quad \forall \vec{x}, \vec{y} \in W$.

(3) $a\vec{x} \in W, \quad \forall a \in F, \quad \vec{x} \in W$.

We will consider F is complex or real field at the same time and let $W = u_2$.

Firstly, $\vec{0}_V$ is zero matrix, and we can easily find

$$\vec{0}_V + \vec{0}_V^* = \vec{0}_V$$

The above identity is true for real and complex field.

Secondly, for $\forall$

$$\vec{x} = \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} \text{and} \quad \vec{y} = \begin{bmatrix} u_2 & v_2 \\ w_2 & z_2 \end{bmatrix} \in W$$

we have

$$\begin{aligned}
(\vec{x}+\vec{y})^* &= \left(\begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} + \begin{bmatrix} u_2 & v_2 \\ w_2 & z_2 \end{bmatrix}\right)^* \\
&= \begin{bmatrix} u_1+u_2 & v_1+v_2 \\ w_1+w_2 & z_1+z_2 \end{bmatrix}^* \\
&= \begin{bmatrix} \overline{u_1+u_2} & \overline{w_1+w_2} \\ \overline{v_1+v_2} & \overline{z_1+z_2} \end{bmatrix} \\
&= \begin{bmatrix} \overline{u_1}+\overline{u_2} & \overline{w_1}+\overline{w_2} \\ \overline{v_1}+\overline{v_2} & \overline{z_1}+\overline{z_2} \end{bmatrix} \\
&= \begin{bmatrix} \overline{u_1} & \overline{w_1} \\ \overline{v_1} & \overline{z_1} \end{bmatrix} + \begin{bmatrix} \overline{u_2} & \overline{w_2} \\ \overline{v_2} & \overline{z_2} \end{bmatrix} \\
&= \vec{x}^* + \vec{y}^* \\
&= -(\vec{x}+\vec{y})
\end{aligned}$$

Therefore we get $(\vec{x}+\vec{y})^* = -(\vec{x}+\vec{y})$ which means condition (2) is true for complex and real field.

In the end, let $a \in F$ and $\vec{x}$ is as previous one. We have

$$\begin{aligned}
(a\vec{x})^* &= \left(\begin{bmatrix} au_1 & av_1 \\ aw_1 & az_1 \end{bmatrix}\right)^* \\
&= \begin{bmatrix} \overline{au_1} & \overline{aw_1} \\ \overline{av_1} & \overline{az_1} \end{bmatrix} \\
&= \overline{a}\begin{bmatrix} \overline{u_1} & \overline{w_1} \\ \overline{v_1} & \overline{z_1} \end{bmatrix} \\
&= -\overline{a}\vec{x}
\end{aligned}$$

We can easily find if F is real field then $(a\vec{x})^* = -(\overline{a}\vec{x})$. However, if F is complex field, $(a\vec{x})^*$ and $-(\overline{a}\vec{x})$ are not necessary same (e.g. $a = i$). To sum up, u_2 is a real subspace of V instead of a complex field.

2. Let

$$\begin{aligned}
U &= \left\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : A^2 = \mathrm{tr}(A)A\right\} \\
V &= \left\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : A^2 + \det(A)I = 0\right\}
\end{aligned}$$

(a) Is U a vector subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

(b) Is V a vector subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

Solution:

To verify U is a real subspace of V, we should consider the 3 conditions:

(1) $\vec{0}_V \in U$.

(2) $\vec{x}+\vec{y} \in U, \quad \forall \vec{x}, \vec{y} \in U$.

(3) $a\vec{x} \in U, \quad \forall a \in F, \quad \vec{x} \in U$.

Firstly, $\vec{0}_V$ is zero matrix, and we can easily find

$$\vec{0}_V^2 = \mathrm{tr}(\vec{0}_V)\vec{0}_V$$

At the very beginning, our normal idea is to verify these three conditions. However when we do the second condition, we may encounter some trouble. Thus we try it in a different way later.

Secondly, for $\forall$

$$\vec{x} = \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} \in U$$

we have

$$\begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} = (u_1+z_1)\begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix}$$

which can be simplified into

$$\begin{bmatrix} v_1 w_1 & v_1 z_1 \\ w_1 u_1 & w_1 v_1 \end{bmatrix} = \begin{bmatrix} z_1 u_1 & z_1^2 \\ u_1^2 & u_1 z_1 \end{bmatrix}$$

There may be many cases. For example, we can have

$$z_1 = v_1, w_1 = u_1, v_1 w_1 = z_1 u_1$$

or

$$z_1 = 0, v_1 = 0$$

It seems very complicated now, but please remember what do we need to prove. If we combine the special form of elements from U and the condition (b), we may feel that it's not closed for addition. For example, we have

$$A = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \text{and } B = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \in U$$

Let $C = A + B$, and then

$$C = \begin{bmatrix} 0 & 1 \\ -1+i & 1 \end{bmatrix}$$

It's easy to see C is not in U. As a result, U is not a subspace.

(b) First we check the form of elements from V.

for $\forall$

$$\vec{x} = \begin{bmatrix} u_1 & v_1 \\ w_1 & z_1 \end{bmatrix} \in U$$

we have

$$\begin{bmatrix} u_1^2 + v_1 w_1 & u_1 v_1 + v_1 z_1 \\ w_1 u_1 + z_1 w_1 & z_1^2 + v_1 w_1 \end{bmatrix} + \begin{bmatrix} u_1 z_1 - v_1 w_1 & 0 \\ 0 & u_1 z_1 - v_1 w_1 \end{bmatrix} = 0$$

which can be simplified into

$$\begin{bmatrix} u_1(u_1 + z_1) & v_1(u_1 + z_1) \\ w_1(u_1 + z_1) & z_1(u_1 + z_1) \end{bmatrix} = 0$$

From above identity, we have either $\vec{x}$ is zero matrix or $\vec{x}$ has opposite diagonal elements which is just $u_1 = -z_1$. Then for any two elements from V, the addition of them is still in V because the sum of their diagonal elements still satisfy the above property. The condition (a) and (c) are easy to verify. Try them on your own.

In the end, we get V is subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.